

# On D. Tymoczko's critique of Mazzola's counterpoint theory

Octavio Alberto Agustín-Aquino  
Universidad de la Cañada

Guerino Mazzola  
University of Minnesota, School of Music and  
University of Zürich, Institut für Informatik

June 14, 2011

## Abstract

Some questions and criticism [11] by D. Tymoczko about Mazzola's counterpoint theory are answered.

## 1 Introduction

Once a writer was asked for the population of Mexico City by a German citizen. When the writer answered “it is about 6 million people, a million more, a million less”, his interlocutor replied “But a million is about the population of my hometown!”. Perhaps as our German friend, we could wish to know the aforementioned figure down to every single individual, but determining this number is an extremely complex labor. For some purposes, it is good enough to count with some approximation, but for many others we need more powerful mathematical tools in order to refine our knowledge.

We can say the same about Tymoczko's worries concerning the sophistication of some mathematical approaches used by Mazzola to delve into the intricacies of musical phenomena. From his examination of Mazzola's *The Topos of Music*, he concludes [11] that Mazzola is tacitly asserting “If you cannot learn algebraic geometry [...], then you have no business trying to understand Mozart”. Such a phrase is similar to “If you cannot learn probability and statistics, then you have no business trying to count the population of a city”. Of course you can undertake such a task without having a hint of mathematics beyond mere arithmetic, but the quality of your results would be accordingly limited. Reality (which includes Mozart's music) is not necessarily simple and easy to grasp, and we can expect that mathematical tools required to increase our understanding of it to be of the same caliber; in particular, in view of Mozart being considered

as a genius. With this in mind, we proceed to reply to some of Tymoczko’s statements and questions.

## 2 The reduced strict style

We should start by noting that Mazzola’s counterpoint model does not use any algebraic geometry. The model only works with the dual numbers ( $\mathbb{Z}_{12}[\epsilon] = \mathbb{Z}_{12}[X]/(X^2)$ ,  $\epsilon$  being the class of  $X$ ) which are in fact used in algebraic geometry, but their structure does not use anything of that theory. Tymoczko has not read the theory with due precision. However, for the classification of local and global compositions, one needs algebraic geometry, in particular existence theorems of quotient schemes modulo actions of finite groups. And it is here that Tymoczko—to avoid perspectives of plagiarism—would better learn a minimum of algebraic geometry, since he was informed in public during the first conference of the Society for Computation and Mathematics in Music in Berlin 2007 by Mazzola that his orbifold approach to chords as quotients of tone sequences modulo a permutation group had already been realized in Mazzola’s local classification theorem in [5, pp. 12, 13].

Tymoczko begins by questioning the simplifications made by Mazzola and Muzzolini in the analysis of the so-called *reduced strict style*, which results from “reducing” modulo octave the Fux’s rules in any possible diatonic scale. For the question “why Mazzola permits progressions [within the reduced strict style] from unisons to octaves (or vice versa), while prohibiting the analogous progressions between fifths and twelfths?” the argument given in the paper [7, p. 19] is: Twelfths are not allowed because major tenths are the maximal intervals admitted; the reference to this restriction is Ernst Tittel’s classical counterpoint book [10, p. 55]. Therefore, by definition of the reduced style, fifths are the only inverse images of the reduction modulo octave and thus parallel fifths are certainly forbidden in the reduced style. But octaves and primes are both allowed and therefore their identification modulo octave cannot result in a forbidden parallel of primes in the reduced style. Again, Tymoczko seems not having read that argument, or perhaps his reading was not precise. It must be stressed that, while in the reduced strict style the parallel fifths are *axiomatically* forbidden, in Mazzola’s model this is *deduced* as a theorem [6, Table O.1.6].

Regarding the “battuta”, the use or prohibition of it is certainly a subtle issue, but there are at least two reasons for not allowing it in the reduced strict style. The first is that Aloysius considers it as an error in his pupil’s exercises (even if he leaves to his discretion to follow the rule or not; in any case, we can choose to follow any rules we wish); the second is that Aloysius himself cannot give a full explanation of the origin of the rule, but the mathematical model can. Specifically, Fux states that the example  $(E3, C4) \rightarrow (D3, D4)$  is acceptable but  $(C3, E4) \rightarrow (D3, D4)$  is not, exactly as Mazzola’s model predicts.

Something similar applies to repetitions; the phrasing of the four “fundamental rules” in Fux’s treatise imply that he does not consider the repetition of a note as an skip or step. In particular, he describes oblique motion as “when

one part moves by step or skip while the other remains stationary” [4]. We should also stress that in Mazzola’s model the repetition of an interval is not forbidden: it is simply not considered, for it only covers changes of intervals. Hence this point of the model has been misunderstood.

### 3 Real-world practice

Tymoczko provides further evidence for his claims (concerning wrong predictions by Mazzola’s model) using examples from actual musical compositions from the Renaissance. Nevertheless, there is a problem exhibiting “deviations” from Mazzola’s rules in the practice of Fux, Lassus or Palestrina, because these amounts for just a few examples or anecdotal evidence. He should show a statistical account of how many times that kind of “violation” occurs in reality. In any case, practice also deviates significantly from Fux’s theory (like using fourths or writing parallel fifths) but this fact is not mentioned by him. In an analysis [9] of first-species passages (with a total of 203 transitions) of the Missa Papae Marcelli by G. P. da Palestrina, the composer infringes on 15 occasions Fux’s rules, and in 6 times Mazzola’s rules. Even if marginally better (considering that Mazzola’s model is nearly as twice more permissive than Fux’s theory), the fit of Mazzola’s theory provides some justification of its relevance. Furthermore, Fux’s theory is supposed to be a reasonable approximation of the practice of counterpoint, so we would expect a closer observance by Palestrina of Fux’s rules than of Mazzola’s model. The comparison we had drawn suggests that neither of the two theories is better in practice.

We can elaborate on the previous point. In the first place, approximately 13% of diatonic transitions are forbidden by Mazzola’s model, whereas roughly 3% of the transitions used by Palestrina in the Missa Papae Marcelli correspond to Mazzola’s prohibitions. Analogously, 22.65% of the diatonic progressions are not acceptable in the reduced strict style, but Palestrina diverged from it 7.39% of the time. This fact can be interpreted in two ways: One is that Tymoczko is right and only a minority of Fux’s rules are reflected in Mazzola’s model; the other is that Palestrina (as a natural process) incurred in Mazzola’s prohibitions in a *smaller* proportion as he did with Fux’s prohibitions. In second place, 65.15% of the diatonic transitions are explained by one or two symmetries<sup>1</sup>, but in Palestrina’s writing this happens in 82.27% of the transitions! This remarkable phenomenon provides further evidence for the musical “anthropic principle” postulated by Mazzola, since Palestrina (of course, unbeknownst to him) preferred the transitions governed by few symmetries more often than statistically expected.

The fact that Tymoczko cannot “imagine a plausible mechanism that would serve to connect Mazzola’s mathematics to real-world musical practice” does not mean that such a connection does not exist. The rigidity of the set of

---

<sup>1</sup>An admissible progression in Mazzola’s model can be explained by one up to five symmetries. For instance, the progression  $(D3, F3) \rightarrow (C3, G3)$  is explained by exactly one counterpoint symmetry.

counterpoint consonances (which were chosen according to musical practice) modulo octave is a mathematical fact, and practically all of the conclusions of Mazzola's model result from it. Tymoczko never refers to this part of the model, except that he claims that the symmetries of the model are "deep", which is by no means true for anyone who has ever studied mathematical group theory. He also contradicts "real-world musical practice" with his own use of the abstract symmetric group in his orbifold theory. Mathematics is always a foundation of "real-worlds", in physics and elsewhere.

Tymoczko also omits the EEG experiments conducted by Mazzola, Wieser et al. [8] concerning the neurophysiology of the consonance/dissonance dichotomy and its polarity, which provide evidence that the human brain reacts to the effect of the polarity on a consonance. Again, music theory and music itself is the creation of human beings. What does "reality" means in such a context?

Additionally, the practical dimension of music can be used in multiple ways to support music theory. Reasons from vocal music may have inspired the prohibitions of tritones and parallel fifths, but this argument is not convincing enough since instrumental counterpoint is not limited by these constraints. In this regard, the intuition provided by what Western humans can sing does not tells us whether there are alternative ways of understanding consonance and dissonance or how to extend counterpoint to the microtonal setup. Where intuition fails, mathematics can suggest the path to follow, with the prediction of six distinct counterpoint "worlds" and explicit rules for microtonal counterpoint in Mazzola's model. These extensions have also been implemented in computer programs by one of the authors [1] and Julien Junod [3]. This is real-world: It sounds, it can be calculated concretely, and its is relevant to composers. Why is it also a real-world concern of theorists? Because the existing counterpoint theory has a weak theoretical basis. Above all, the forbidden parallels of fifths are justified in a psychoacoustic way: perfect intervals are near to fuse. But this is a justification which refers to another ontology, which is not music-theoretical, but psychological. The reference is to just tuning (perfect fifth has frequency ratio  $3/2$ ), but the same argumentation would classify the perfect fourth (with frequency ratio  $4/3$ ) as being consonant; in fact, this is true in Pythagorean music theory. Carl Dahlhaus [2] has pointed out that it is not psychoacoustics, but compositional theory which could explain the dissonant qualification of the fourth in counterpoint. Our model, refraining from reference to psychoacoustics, gives an answer regarding this problem, and just tuning only creates the contradiction with the perfect fourth as explained above. Our reference to pitch classes is not, as Tymoczko believes, our anachronistic attitude, but a simple mathematical abstraction without diachronic relevance. Embedding a diatonic scale into a chromatic one is not, as it is suggested, a special property of pitch classes; this is well known to be the case for just tuning, too.

It is interesting that the blending of perfect with imperfect consonances, which is generally accepted as a core theme arised from counterpoint, is not seen by Tymoczko as being dealt with and perfectly modeled with the local symmetries in our model. The guidance we provide for moving between these consonances is a direct consequence of these symmetries. They "internalize"

the external dichotomy consonances/dissonances within the consonant half of intervals. The use of local symmetries is explicitly related to their well known role in the creation of tension in physics, so the argument of describing tension by local symmetries is clearly given. And we shall insist: All the allowed and forbidden interval transitions are derived from the model and they are nowhere presupposed. Tymoczko does not understand this difference, and he just repeats the psychoacoustic argument for forbidden parallels of fifths.

With respect to the interchangeability of voices, there are ways of discerning cantus firmus from discantus; in some compositions the difference is obvious because of the appearance of fourths. In other words: if the distinction between cantus firmus and discantus were pointless, double counterpoint would be trivial. Thus the asymmetry between cantus firmus and discantus is essential for counterpoint, even if it is a necessary hypothesis for an orbifold model. Besides, Mazzola’s theory provides a precise mathematical manipulation that permits voice crossings [6, Section 29.6]. Tymoczko cannot cover the distinction between sweeping and hanging counterpoint, for only a hierarchy of cantus firmus against discantus can do this. The intervals in the dual number description are therefore associated with two projections  $\alpha_{pm} : \mathbb{Z}_{12}[\epsilon] \rightarrow \mathbb{Z}_{12} : a + \epsilon.b \mapsto a \pm b$ , addition for sweeping, subtraction for hanging counterpoint. In hanging counterpoint  $a - 7$  is allowed, but in sweeping counterpoint the same pitch class  $a + 5$  is not; these subtleties escape the unordered model.

## 4 Conclusion

Finally, some remarks concerning the “statistical rebuttal” are in order. Let us accept Tymoczko’s count of 65 Fux-inadmissible diatonic progressions among the 287 possible, and that only 19 of them are correctly predicted by Mazzola’s model. The probability of guessing 19 *or more* prohibitions in 37 trials is

$$\frac{1}{\binom{287}{37}} \sum_{k=19}^{37} \binom{65}{k} \binom{222}{37-k} \approx 3.6023 \times 10^{-5},$$

or less than one in twenty thousands. Thus in 99.996% of a large number of experiments involving a person trying to guess counterpoint rules in 37 trials (with no knowledge of the technique), the individual will stumble upon with 18 *or less* prohibitions. Moreover, his average number of matches would be between 8 and 9.

Considering the number of mathematical models of counterpoint that have been formulated during the last two millenia (which, for sure, are much less than twenty thousands), could we think that one that makes a 29% of correct predictions is bad taking into account that we would expect only a 13% success rate on average? It is true that the mathematical model’s predictions are not random, but they could be thought as such since they do not rely on the psychoacoustic or vocal considerations that, allegedly, led to Fux’s theory. Note we call a success a concordance with *another* theory, not with actual musical

practice. In fact, an agreement close to 100% would be rather suspicious (in particular if the model is almost as complex as the phenomenon we are trying to explain) for it could be an example of overfitting.

## References

- [1] O. A. Agustín-Aquino. Counterpoint in  $2k$ -tone equal temperament. *Journal of Mathematics and Music*, 3(3):153–164, 2009.
- [2] C. Dahlhaus. Zur Theorie des klassischen Kontrapunkts. *Kirchenmusikalisches Jb*, 45:43–57, 1961.
- [3] J. Junod. *Counterpoint Worlds and Morphisms*. PhD thesis, Universität Zürich, 2010.
- [4] A. Mann. *The Study of Counterpoint*. W. W. Norton & Company, 1965. Translations of fragments of the *Gradus ad Parnassum* by J. J. Fux.
- [5] G. Mazzola. *Gruppen und Kategorien in der Musik*. Heldermann, Berlin, 1985.
- [6] G. Mazzola. *The Topos of Music*. Birkhäuser, 2002.
- [7] G. Mazzola and D. Muzzolini. Deduktion des Quintparallelenverbots aus der Konsonanz-Dissonanz-Dichotomie. Accepted for publication in *Musiktheorie*, Laaber, 1990.
- [8] G. Mazzola, H. G. Wieser, F. Brunner, and D. Muzzolini. A symmetry-oriented mathematical model of counterpoint and related neurophysiological investigations by depth-EEG. In I. Hargittai, editor, *Symmetry: Unifying Human Understanding*. CAMWA/Pergamon Press, New York, 1989.
- [9] A. Nieto. Una aplicación del teorema de contrapunto. Tesis de Licenciatura, ITAM, 2010.
- [10] E. Tittel. *Der neue Gradus*. Doblinger, Wien-München, 1959.
- [11] D. Tymoczko. *Mazzola’s counterpoint theory*, 2011.